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On the fixed-point set of automorphisms of non-orientable surfaces without boundary

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Abstract Macbeath gave a formula for the number of fixed points for each non-identity element of a cyclic group of automorphisms of a compact Riemann surface in terms of the universal covering transformation group of the cyclic group. We observe that this formula generalizes to determine the fixed-point set of each non-identity element of a cyclic group of automorphisms acting on a closed non-orientable surface with one exception; namely, when this element has order 2. In this case the fixed-point set may have simple closed curves (called *ovals*) as well as fixed points. In this note we extend Macbeath's results to include the number of ovals and also determine whether they are twisted or not.

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For David Epstein on the occasion of his sixtieth birthday

1 Introduction

Let Y be a compact non-orientable Klein surface of genus $p \geq 3$. By genus here we mean the number of cross-caps of the surface. Let $t\colon Y \to Y$ be an automorphism of order M. If $1 \leq i < M$ and if $i \neq M/2$ then the fixed-point set of t^i consists of isolated fixed points and their number can be calculated, as described below, by a formula which is completely analogous to Macbeath's formula [5] concerning automorphisms of Riemann surfaces. However, if $M = 2N$ then the fixed-point set of the involution t^N consists of a finite number n of disjoint simple closed curves called *ovals* together with a finite number of isolated fixed points [2], [6]. The ovals may be *twisted* or *untwisted* which means that they have Möbius band or annular neigbourhoods respectively.

In this note we calculate the number of ovals and isolated fixed-points of t^N and whether the ovals are twisted or not.

The information is given, as in Macbeath [5] in terms of the universal covering transformation group.

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2 The universal covering transformation group

If Y is a compact non-orientable Klein surface of genus $p \geq 3$ then the orientable two-sheeted covering surface of Y has genus ≥ 2, so that the universal covering space of Y is the upper half-plane H (with the hyperbolic metric) and the group of covering transformations is a non-orientable surface subgroup K generated by glide-reflections. If G is a group of automorphisms of Y then the elements of G lift to a *non-euclidean crystallographic (NEC) group* Γ acting on H. There is a smooth epimorphism

$$\theta\colon \Gamma \to G \tag{1}$$

whose kernel is K, where smooth means that θ preserves the orders of elements of finite order in Γ. The transformation group $(\Gamma, \mathcal{H})$ is called the *universal covering transformation group* of (G, Y).

Now let $G = \langle t | t^{2N} = 1\rangle$ be a cyclic group of order $2N$. As θ is smooth we must have $\theta(c) = t^N$ for every reflection c in Γ. Also we cannot have two distinct reflections in Γ whose product has finite order. So it follows, in the canonical presentation of NEC groups as given in [4] or [3], that Γ has empty period cycles.

Thus Γ has signature of the form

$$s(\Gamma) = (g; \pm; [m_1, ..., m_n]; \{(\quad)^k\}) \tag{2}$$

with k empty period cycles; then Γ has one of the two presentations depending on whether there is a $+$ or a $-$ in the signature;

for the $(+)$ case

$$\begin{gathered} x_1, \ldots, x_n, e_1, \ldots, e_k, c_1, \ldots, c_k, a_1, b_1, \ldots, a_g, b_g \mid \\ x_i^{m_i} = 1, i = 1, ..., n, c_j^2 = c_j e_j^{-1} c_j e_j = 1, j = 1, ..., k, \\ x_1 ... x_n e_1 ... e_k a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a g h^{-1} b_g^{-1} \end{gathered} \tag{3}$$

for the $(-)$ case

$$x_1, \ldots, x_n, e_1, \ldots, e_k, c_1, \ldots, c_k, d_1, ..., d_g \mid$$
$$x_i^{m_i} = 1, i = 1, ..., n, c_j^2 = c_j e_j^{-1} c_j e_j = 1, j = 1, ..., k, x_1 ... x_n e_1 ... e_k d_1^2 ... d_g^2 \quad (4)$$

In these presentations the generators x_i are elliptic elements, the generators c_j are reflections, *the generating reflections* of Γ, and the generators e_j are orientation-preserving transformations called the *connecting generators*. Each empty period cycle corresponds to a conjugacy class of reflections in Γ.

One important fact to note about these presentations is that the connecting generator e_j commutes with the generating reflection c_j, and in fact the centralizer of c_j in Γ is just the group $gp\langle c_j,\ e_j\rangle \cong C_2 \times C_\infty$. (See [8])

3 The fixed-point set of a power of t

Let Y be a non-orientable surface of topological genus $p \geq 3$ and let t be an automorphism of order $2N$. If $1 \leq i < 2N$ and $i \neq N$ then the number of fixed points of the automorphism t^i is given by Macbeath's formula (see [5]). If t^i has order d than t^i has

$$2N \sum_{d|m_j} \frac{1}{m_j} \quad (5)$$

fixed points, where m_j runs over the periods in $s(\Gamma)$.

This is because Macbeath's proof (applying to Fuchsian groups) only uses the facts that each period corresponds to a unique conjugacy class of elliptic elements of Γ, and each elliptic element has a unique fixed point in H. Now, the number of isolated fixed points of t^i is independent of the smooth epimorphism θ above. However the epimorphism θ does play a part in the number of ovals of t^N.

Theorem 3.1 *Let Y be a non-orientable surface of topological genus $p \geq 3$. Let $G \cong C_{2N} = \langle t \mid t^{2N} = 1\rangle$ be a group of automorphisms of Y, and let θ and Γ be as described in equations 1 and 2. If $\theta(e_j) = t^{v_j}$ than the number of ovals of the involution t^N is*

$$\sum_{j=1}^{k} (N, v_j) \quad (6)$$

and the number of isolated fixed points of t^N is

$$2N \sum_{m_j\ even} \frac{1}{m_j}.$$

Proof Let $\Lambda = \theta^{-1}(\langle t^N \rangle)$ so that Λ contains the group $K = Ker\theta$ with index 2. Now, Λ must have signature of the form

$$s(\Gamma) = (g; \pm; [2^{(r)}]; \{(\quad)^s\}) \tag{7}$$

with r periods equal to 2 and s empty period cycles.

The reason that all periods in Λ are equal to 2 is because if m_j in $s(\Gamma)$ is even then $x_j^{m_j/2} \in \Lambda$ and any elliptic element of Λ are conjugate to some $x_j^{m_j/2}$ (see [7]).

By results in [2] (see also [3]), r is the number of isolated fixed points of t^N and is given by Macbeath's formula

$$2N \sum_{m_j \ even} \frac{1}{m_j}$$

It also follows from [2] that the number of ovals of t^N is just the number s of period cycles in Λ, which corresponds to the number of conjugacy classes of reflections in Λ. As a reflection c_j in Λ belongs also to Γ and the group Γ has k conjugacy classes of reflections, we just have to determine into how many Λ–conjugacy classes the Γ–conjugacy class of c_j splits. We shall use the epimorphism θ to calculate this number.

There is a transitive action of Γ on the Λ–conjugacy classes of c_j in Λ by letting $\gamma \in \Gamma$ map the reflection gc_jg^{-1} to $g\gamma c_j \gamma^{-1} g^{-1}$, with $g \in \Lambda$. (Because $\Lambda \triangleleft \Gamma$). Clearly, if $\lambda \in \Lambda$ then λ has a trivial action on these Λ–conjugacy classes. So we have an action of $\Gamma/\Lambda \cong C_{2N}/C_2 \cong C_N$ on these classes. As the centralizer of c_j in Γ is just $\langle c_j, e_j \rangle$, the stabilizer of the Λ–conjugacy classes of c_j in Λ are the cosets $\Lambda, \Lambda e_j, \ldots, \Lambda e_j^{\delta_j - 1}$, where $\delta_j = exp_\Lambda e_j$, the least positive power of e_j that belongs to Λ. Now, let $\varepsilon_j = exp_K e_j$. Then either $\varepsilon_j = \delta_j$ or $\varepsilon_j = 2\delta_j$.

The additive group Z_{2N} contains a subgroup isomorphic to Z_N and $a \in Z_N$ has order $\frac{N}{(N,a)}$ in Z_N so that a has the same order in Z_{2N} if and only if $(2N, a) = 2(N, a)$. If $(2N, a) = (N, a)$ then the order of a in Z_{2N} is twice the order of a in Z_N and we then find that

$$\varepsilon_j = \delta_j \qquad \text{if} \qquad (2N, v_j) = 2(N, v_j)$$

and

$$\varepsilon_j = 2\delta_j \qquad \text{if} \qquad (2N, v_j) = (N, v_j),$$

where $\theta(e_j) = t^{v_j}$.

By the above argument on the action of Γ/Λ on the Λ–conjugacy classes of c_j we see that the number of such classes is N/δ_j, which is

if $\varepsilon_j = \delta_j$

$$\frac{N}{\delta_j} \quad = \quad \frac{N}{\varepsilon_j} \quad = \quad \frac{N(2N, v_j)}{2N} \quad = \quad \frac{(2N, v_j)}{2} \quad = \quad (N, v_j),$$

or if $\varepsilon_j = 2\delta_j$

$$\frac{N}{\delta_j} \quad = \quad \frac{2N}{\varepsilon_j} \quad = \quad \frac{2N(2N, v_j)}{2N} \quad = \quad (2N, v_j) \quad = \quad (N, v_j)$$

Thus in both cases the generating reflection c_j of Γ induces (N, v_j) conjugacy classes of reflections in Λ. Thus the number of ovals of t^N in Y is

$$\sum_{j=1}^{k} (N, v_j) \tag{8}$$

□

Theorem 3.2 *The ovals of t^N in Y induced by the jth period cycle in Γ are twisted if $(2N, v_j) = (N, v_j)$ and untwisted if $(2N, v_j) = 2(N, v_j)$.*

Proof As we have found in Theorem 3.1, the jth empty period cycle in Γ induces (N, v_j) empty period cycles in Λ. The generating reflections of these period cycles are just conjugates of c_j in Γ and, as the corresponding connecting generator e_j is just the orientation-preserving element generating the centralizer of c_j in Γ, we see that the connecting generator of each of the period cycles in Λ induced by the jth period cycle in Γ is just conjugate to $e_j^{\delta_j}$, $\delta_j = exp_\Lambda e_j$ as in the proof of Theorem 3.1. Now, let $\theta'\colon \Lambda \to C_2 = gp\langle\xi\rangle$, where $\xi = t^N$, be the restriction of the epimorphism $\theta\colon \Gamma \to C_{2N}$. Then

if $\varepsilon_j = \delta_j$

$$\theta'(e_j^{\delta_j}) \quad = \quad \theta'(e_j^{\varepsilon_j}) \quad = \quad \theta(e_j^{\varepsilon_j}) \quad = \quad 1$$

if $\varepsilon_j = 2\delta_j$

$$\theta'(e_j^{\delta_j}) \quad = \quad \theta'(e_j^{\frac{\varepsilon_j}{2}}) \quad = \quad \theta(e_j^{\frac{\varepsilon_j}{2}}) \quad = \quad \xi,$$

ξ the generator of C_2. Generally, if c is the generating reflection of an empty period cycle of Λ and e is the corresponding connecting generator then figures 1 and 2 show that $\theta'(e) = 1$ corresponds to an untwisted oval while $\theta'(e) = \xi$ corresponds to a twisted oval.

However, as in the proof of Theorem 3.1 $\varepsilon_j = \delta_j$ if and only if $(2N, v_j) = 2(N, v_j)$ and hence we have untwisted ovals while $\varepsilon_j = 2\delta_j$ if and only if $(2N, v_j) = (N, v_j)$ and we have twisted ovals. □

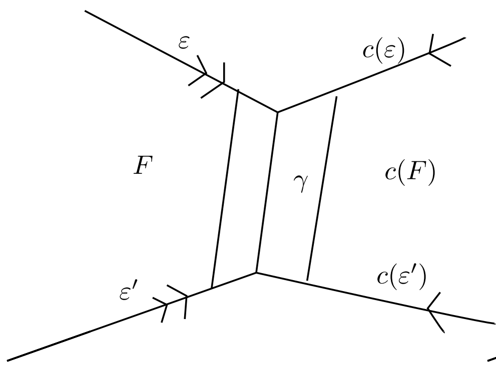

Figure 1: $\theta'(e) = 1$ so $e \in K$

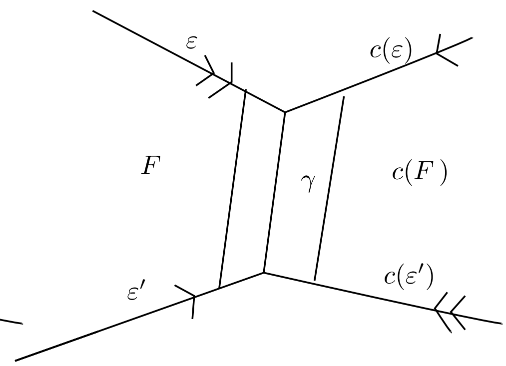

Figure 2: $\theta'(e) = \xi$ so $ce \in K$

4 Bounds and examples

In [6] (also see [2]) Scherrer showed that that if an involution of a non-orientable surface of genus p has $\mid F \mid$ fixed points and $\mid V \mid$ ovals then

$$\mid F \mid + 2 \mid V \mid \leq p + 2.$$

In our examples we will show that for any integer N we can find a non-orientable surface of genus p admitting a C_{2N} action with generator t such that t^N attains the Scherrer bound.

Example 1 Bujalance [1] found the maximum order for an automorphism t of a non-orientable surface Y of genus $p \geq 3$; it is $2p$ for odd p and $2(p-1)$ for even p. The universal covering transformation group Γ has signature $s(\Gamma) = (0; [2, p]; \{(\quad)\})$ for odd p, and signature $s(\Gamma) = (0; [2, 2(p-1)]; \{(\quad)\})$ for even p. There is, essentially, only one way of defining the epimorphism θ in each case:

if p is odd, we define $\theta\colon \Gamma \to C_{2p}$ by $\theta(x_1) = t^p$, $\theta(x_2) = t^2$, $\theta(c) = t^p$, and $\theta(e) = t^{p-2}$,

if p is even, we define $\theta\colon \Gamma \to C_{2(p-1)}$ by $\theta(x_1) = t^{p-1}$, $\theta(x_2) = t^1$, $\theta(c) = t^{p-1}$, and $\theta(e) = t^{p-2}$.

Using Macbeath's formula (5) we see that the involution t^p has p fixed points for surfaces of both odd and even genera. Now, if p is odd then the involution t^p also has, by Theorems 3.1 and 3.2, one twisted oval if p is odd as $(p, p-2) = (2p, p-2) = 1$. If p is even then the involution t^{p-1} has, by Theorems 3.1 and 3.2, one untwisted oval as $(p-1, p-2) = 1$ and $(2(p-1), p-2) = 2(p, p-2) = 2$. We note that the involution t^p obeys the Scherrer bound. Note that the orders

of the cyclic groups in Bujulance's examples are $\equiv 2 \bmod 4$. Our second example shows that the Scherrer bound can be obtained for the involution in a C_4 action.

Example 2 Let Y be a non-orientable surface of genus $p \geq 3$, and let t be an automorphism of Y of order 4. Let Γ have signature

$$(0; +; [2^{(r)}, 4, 4]; (\quad)^k)$$

and define a smooth epimorphism $\theta\colon \Gamma \to C_4$ by mapping the generators of order two to t^2, the two generators of order 4 to t and t^{-1} and the connecting generators to the identity. We then find that for the involution t^2, $\mid F \mid = 2r+2$, and $\mid V \mid = 2k$,and $p = 4k + 2r$, so that we find infinitely many surfaces where the Scherrer bound is attained for the involution in C_4. This is easily extended to groups of order $4m$ by replacing the two periods 4 in the signature of Γ by $4m$.

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